EXTREMAL METRICS AND LOWER BOUND OF THE MODIFIED K-ENERGY

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Abstract. We provide a new proof of a result of X.X. Chen and G.Tian [4]: for a polarized extremal Kähler manifold, an extremal metric attains the minimum of the modified K-energy. The proof uses an idea of Chi Li [16] adapted to the extremal metrics using some weighted balanced metrics.

1. Introduction

Extremal metrics were introduced by Calabi [1]. Let (X, ω) be a Kähler manifold of complex dimension n. An extremal metric is a critical point of the functional

$$g \quad \mapsto \quad \int_X (S(g))^2 \frac{\omega_g^n}{n!}$$

defined on Kähler metrics g representing the Kähler class $[\omega]$, where $S(g)$ is the scalar curvature of the metric g. Constant scalar curvature Kähler metrics (CSCK for short), and in particular Kähler-Einstein metrics, are extremal metrics. In this work we will focus on the polarized case, assuming that there is an ample holomorphic line bundle $L \to X$ with $c_1(L) = [\omega]$. In this special case, it has been conjectured by Yau in the Kähler-Einstein case [29], and then in the CSCK case by the work of Tian [27] and Donaldson [9] that the existence of a CSCK metric representing $c_1(L)$ should be equivalent to a GIT stability of the pair (X, L). This conjecture has been extended to extremal metrics by Székelyhidi [25] and Mabuchi [20].

Let (X, L) be a polarized Kähler manifold. Donaldson has shown [8] that if X admits a CSCK metric in $c_1(L)$, and if $\mathrm{Aut}(X, L)$ is discrete, then the CSCK metric can be approximated by a sequence of balanced metrics. This approximation result implies in particular the unicity of a CSCK metric in its Kähler class. This method has been adapted by Mabuchi [19] to the extremal metric setting to prove unicity of an extremal metric up to automorphisms in a polarized Kähler class. Then, Chen and Tian proved unicity of an extremal metric in its Kähler class up to automorphisms with no polarization assumption [4].

In a sequel to his work on balanced metrics [10], Donaldson shows that if $\mathrm{Aut}(X, L)$ is discrete, a CSCK metric is an absolute minimum of the Mabuchi energy E, or K-energy, introduced by Mabuchi [18]. The approximation result of Donaldson does not hold true for CSCK metrics if the automorphism group is not discrete. There are counter-examples of Ono, Yotsutani and the first author [21], or Della Vedova and Zudas [6]. However, Li managed to show that even if $\mathrm{Aut}(X, L)$ is not discrete, a CSCK metric would provide an absolute minimum of E [16].

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By a theorem of Calabi [2], extremal metrics are invariant under a maximal connected compact sub-group G of the reduced automorphism group $\mathrm{Aut}_0(X)$ [11]. Any two such compact groups are conjugated in $\mathrm{Aut}_0(X)$ and the study of extremal metrics is done modulo one such group. In the extremal setting, the modified K-energy E^G (see definition 2.2.6) plays the role of the K-energy for CSCK metrics. This functional has been introduced independently by Guan [14], Simanca [24] and Chen and Tian [4] and is defined on the space of G-invariant Kähler potentials with respect to a G-invariant metric. In [4], Chen and Tian prove that extremal metrics minimize the modified K-energy up to automorphisms of the manifold, with no polarization assumption. In this paper, we give a different proof of this result in the polarized case. We generalize Li's work to extremal metrics, using some weighted balanced metrics, which are called σ-balanced metrics (see definition 2.2.8 in section 2):

Theorem A. *Let (X, L) be a polarized Kähler manifold and G a maximal connected compact sub-group of the reduced automorphism group $\mathrm{Aut}_0(X)$. Then G-invariant extremal metrics representing $c_1(L)$ attain the minimum of the modified K-energy E^G.*

The proof relies on two observations. We will consider a sequence of Fubini-Study metrics ω_k associated to Kodaira embeddings of X into higher and higher dimension projective spaces. The first observation is that if we define ω_k to be the metric associated to an extremal metric in $c_1(L)$ by the map $Hilb_k$ (see definition in section 2, equation (3)), then ω_k will be close to a σ-balanced metric. The second point is that σ-balanced metrics, if they exist, are minima of the functionals Z_k^σ (section 2, equation (8)) that converge to the modified Mabuchi functional. Then a careful analysis of the convergence properties of the ω_k and Z_k^σ yields the proof of our main result.

Remark 1.0.1. We shall mention that Guan shows in [14] that extremal metrics are local minima, assuming the existence of C^2-geodesics in the space of Kähler potentials.

1.1. Plan of the paper.

In section 2, we review the definition of extremal metrics and recall quantization of CSCK metrics. We then introduce σ-balanced metrics and the relative functionals. Then in section 3, we prove the main theorem. In the Appendix, we collect some facts and proofs of properties of σ-balanced metrics.

1.2. Acknowledgments.

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2. Extremal metrics and Quantization

2.1. Quantization.

Let (X, L) be a polarized Kähler manifold of complex dimension n. Let $\mathcal{H}$ be the space of smooth Kähler potentials with respect to a fixed

Kähler form $\omega \in c_1(L)$:

$$\mathcal{H} = \{\phi \in C^\infty(X) \mid \omega_\phi := \omega + \sqrt{-1}\partial\overline{\partial}\phi > 0\}.$$

For each k, we can consider $\mathcal{H}_k$ the space of hermitian metrics on $L^{\otimes k}$. To each element $h \in \mathcal{H}_k$ one associates a metric $-\sqrt{-1}\partial\overline{\partial}log(h)$ on X, identifying the spaces $\mathcal{H}_k$ to $\mathcal{H}$. Write ω_h to be the curvature of the hermitian metric h on L. Fixing a base metric h_0 in $\mathcal{H}_1$ such that $\omega = \omega_{h_0}$ the correspondence reads

$$\omega_\phi = \omega_{e^{-\phi}h_0} = \omega + \sqrt{-1}\partial\overline{\partial}\phi.$$

We denote by $\mathcal{B}_k$ the space of positive definite Hermitian forms on $H^0(X, L^{\otimes k})$. Let $N_k = dim(H^0(X, L^k))$. The spaces $\mathcal{B}_k$ are identified with $GL_{N_k}(\mathbb{C})/U(N_k)$ using the base metric h_0^k. These symmetric spaces come with metrics d_k defined by Riemannian metrics:

$$(H_1, H_2)_h = Tr(H_1 H^{-1} \cdot H_2 H^{-1}).$$

There are maps :

$$\begin{aligned} Hilb_k : \mathcal{H} &\rightarrow \mathcal{B}_k \\ FS_k : \mathcal{B}_k &\rightarrow \mathcal{H} \end{aligned}$$

defined by :

$$\forall h \in \mathcal{H}\ ,\ s \in H^0(X, L^{\otimes k})\ ,\ ||s||^2_{Hilb_k(h)} = \int_X |s|^2_h d\mu_h$$

and

$$\forall H \in \mathcal{B}_k\ ,\ FS_k(H) = \frac{1}{k}\log\sum_\alpha |s_\alpha|^2_{h_0^k}$$

where $\{s_\alpha\}$ is an orthonormal basis of $H^0(X, L^{\otimes k})$ with respect to H. Note that $\omega_{FS_k(H)}$ is the pull-back of the Fubini-Study metric on $\mathbb{CP}_{N_k-1}$ under the projective embedding induced by $\{s_\alpha\}$. A result of Tian [26] states that any Kähler metric ω_ϕ in $c_1(L)$ can be approximated by projective metrics, namely

$$\lim_{k\to\infty} \frac{1}{k} FS_k \circ Hilb_k(\phi) = \phi$$

where the convergence is uniform on $C^2(X, \mathbb{R})$ bounded subsets of $\mathcal{H}$.
The metrics satisfying

$$FS_k \circ Hilb_k(\phi) = \phi$$

are called balanced metrics, and the existence of such metrics is equivalent to the Chow stability of (X, L^k) by Zhang [31] and Wang [28]. Let $\mathrm{Aut}(X, L)$ be the group of automorphisms of the pair (X, L). From the work of Donaldson [8], if X admits a CSCK metric in the Kähler class $c_1(L)$, and if $\mathrm{Aut}(X, L)$ is discrete, then there are balanced metrics ω_{ϕ_k} for k sufficiently large, with

$$FS_k \circ Hilb_k(\phi_k) = \phi_k$$

and these metrics converge to the CSCK metric on $C^\infty(X, \mathbb{R})$ bounded subsets of $\mathcal{H}$.

In the proof of these results, the density of state function plays a central role. For any $\phi \in \mathcal{H}$ and $k > 0$, let $\{s_\alpha\}$ be an orthonormal basis of $H^0(X, L^k)$ with respect to $Hilb_k(\phi)$. The k^{th} Bergman function of ϕ is defined to be :

$$\rho_k(\phi) = \sum_\alpha |s_\alpha|^2_{h^k}.$$

It is well known that a metric $\phi \in Hilb_k(\mathcal{H})$ is balanced if and only if $\rho_k(\phi)$ is constant. A key result in the study of balanced metrics is the following expansion:

Theorem 2.1.1 ([3],[23],[26],[30])**.** *The following uniform expansion holds*

$$\rho_k(\phi) = k^n + A_1(\phi)k^{n-1} + A_2(\phi)k^{n-2} + ...$$

with $A_1(\phi) = \frac{1}{2}S(\phi)$ *is half of the scalar curvature of the Kähler metric* ω_ϕ *and for any* l *and* $R \in \mathbb{N}$*, there is a constant* $C_{l,R}$ *such that*

$$||\rho_k(\phi) - \sum_{j \le R} A_j k^{n-j}||_{C^l} \le k^{n-R}.$$

As a corollary, if $\phi_k = FS_k \circ Hilb_k(\phi)$, then

$$\phi_k - \phi = \frac{1}{k}\log \rho_k(\phi) \to 0$$

as $k \to \infty$. In particular we have the convergence of metrics

$$\omega_{\phi_k} = \omega_\phi + O(k^{-2}). \tag{1}$$

By integration over X we also deduce

$$\int_X \rho_k(\phi) d\mu_\phi = k^n \int_X d\mu_\phi + k^{n-1}\frac{1}{2}\int_X S(\phi) d\mu_\phi + \mathcal{O}(k^{n-2})$$

where $S(\phi)$ is the scalar curvature of the metric g_ϕ associated to the Kähler form ω_ϕ and $d\mu_\phi = \dfrac{\omega_\phi^n}{n!}$ is the volume form. Thus

$$N_k = k^n V + \frac{1}{2} V \underline{S} k^{n-1} + \mathcal{O}(k^{n-2}). \tag{2}$$

where

$$\underline{S} = 2n\pi \frac{c_1(L) \cup [\omega]^{n-1}}{[\omega]^n}$$

is the average of the scalar curvature and V is the volume of $(X, c_1(L))$.

2.2. **The relative setup.** In order to find a canonical representative of a Kähler class, Calabi suggested [1] to look for minima of the functional

$$\begin{aligned} Ca : \mathcal{H} &\to \mathbb{R} \\ \phi &\mapsto \int_X (S(\phi) - \underline{S})^2 d\mu_\phi. \end{aligned}$$

In fact, critical points for this functional are local minima, called extremal metrics. The associated Euler-Lagrange equation is equivalent to the fact that $grad_{\omega_\phi}(S(\phi))$ is a holomorphic vector field and constant scalar curvature metrics, CSCK for short, are extremal metrics.

By a theorem of Calabi [2], the connected component of identity of the isometry group of an extremal metric is a maximal compact connected subgroup of $\mathrm{Aut}_0(X)$. As all these maximal subgroups are conjugated, the quest for extremal metrics can be done modulo a fixed group action. Note that $\mathrm{Aut}_0(X)$ is isomorphic to

$\mathrm{Aut}_0(X, L)$ the connected component of identity of $\mathrm{Aut}(X, L)$. As we will see later, it will be useful to consider a less restrictive setup, working modulo a circle action. We then define the relevant functionals in a general situation and we fix G a compact subgroup of $\mathrm{Aut}_0(X, L)$ and denote by $\mathfrak{g}$ its Lie algebra.

2.2.1. *Space of potentials.* We extend the quantization tools to the extremal metrics setup.

Replacing L by a sufficiently large tensor power if necessary, we can assume that $\mathrm{Aut}_0(X, L)$ acts on L (see e.g. [15]). Then the G-action on X induces a G-action on the space of sections $H^0(X, L^k)$. This action in turn provides a G-action on the space $\mathcal{B}_k$ of positive definite hermitian forms on $H^0(X, L^k)$ and we define $\mathcal{B}_k^G$ to be the subspace of G-invariant elements. The spaces $\mathcal{B}_k^G$ are totally geodesic in $\mathcal{B}_k$ for the distances d_k. Define $\mathcal{H}^G$ to be the space of G-invariant potentials with respect to a G-invariant base point ω. We see from their definitions that we have the induced maps :

$$\begin{array}{rccc} Hilb_k : & \mathcal{H}^G & \to & \mathcal{B}_k^G \\ FS_k : & \mathcal{B}_k^G & \to & \mathcal{H}^G. \end{array} \tag{3}$$

2.2.2. *Modified K-energy.* For a fixed metric g, we say that a vector field V is a hamiltonian vector field if there is a real valued function f such that

$$V = J\nabla_g f$$

or equivalently

$$\omega(V, \cdot) = -df.$$

For any $\phi \in \mathcal{H}^G$, let P_ϕ^G be the space of normalized (i.e. mean value zero) Killing potentials with respect to g_ϕ whose corresponding hamiltonian vector fields lie in $\mathfrak{g}$ and let Π_ϕ^G be the orthogonal projection from $L^2(X, \mathbb{R})$ to P_ϕ^G given by the inner product on functions

$$(f, g) \mapsto \int fg d\mu_\phi.$$

Note that G-invariant metrics satisfying $S(\phi) - \underline{S} - \Pi_\phi^G S(\phi) = 0$ are extremal.

Definition 2.2.3.[13, Section 4.13] The reduced scalar curvature S^G with respect to G is defined by

$$S^G(\phi) = S(\phi) - \underline{S} - \Pi_\phi^G S(\phi).$$

The extremal vector field V^G with respect to G is defined by the equation

$$V^G = \nabla_g(\Pi_\phi^G S(\phi))$$

for any ϕ in $\mathcal{H}^G$ and does not depend on ϕ (see e.g. [13, Proposition 4.13.1]).

Remark 2.2.4. Note that by definition the extremal vector field is real-holomorphic and lies in $J\mathfrak{g}$ where J is the almost-complex structure of X, while JV^G lies in $\mathfrak{g}$.

Remark 2.2.5. When $G = \{1\}$ we recover the normalized scalar curvature. When G is a maximal compact connected subgroup, or maximal torus of $\mathrm{Aut}_0(X)$, we find the reduced scalar curvature and the usual extremal vector field initially defined by Futaki and Mabuchi [12].

We are now able to define the relative Mabuchi K-energy, introduced by Guan [14], Chen and Tian [4], and Simanca [24]:

Definition 2.2.6.[13, Section 4.13] The modified Mabuchi K-energy E^G (relative to G) is defined, up to a constant, as the primitive of the following one-form on $\mathcal{H}^G$:

$$\phi \mapsto -S^G(\phi)d\mu_\phi.$$

If $\phi \in \mathcal{H}^G$, then the modified K-energy admits the following expression

$$E^G(\phi) = -\int_X \phi(\int_0^1 S^G(t\phi)d\mu_{t\phi}dt).$$

As for CSCK metrics, G-invariant extremal metrics whose extremal vector field lie in $J\mathfrak{g}$ are critical points of the relative Mabuchi energy.

2.2.7. *The σ-balanced metrics.* We present a generalization of balanced metrics adapted to the relative setting of extremal metrics.

Definition 2.2.8. Let $\sigma_k(t)$ be a one-parameter subgroup of $\mathrm{Aut}_0(X, L^k)$. Let $\phi \in \mathcal{H}$. Then ϕ is a k^{th} σ_k-balanced metric if

$$\omega_{kFS_k \circ Hilb_k(\phi)} = \sigma_k(1)^* \omega_{k\phi} \tag{4}$$

Conjecturally, the σ-balanced metrics would provide the generalization of the notion of balanced metric and would approximate an extremal Kähler metric. Indeed, in one direction, assume that we are given σ_k-balanced metrics ω_{ϕ_k}, with $\sigma_k \in \mathrm{Aut}_0(X, L^k)$ such that the ω_k converge to ω_∞. Suppose that the vector fields $k\frac{d}{dt}|_{t=0}\sigma_k(t)$ converge to a vector field $V_\infty \in \mathfrak{h}_0$. A simple calculation implies that ω_∞ must be extremal.

We now define the functionals that play the role of finite dimensional versions of the modified Mabuchi K-energy on $\mathcal{B}_k^G$ and $FS_k(\mathcal{B}_k^G)$ respectively. First define $I_k = \log \circ \det$ on $\mathcal{B}_k^G$. This functional is defined up to an additive constant when we see $\mathcal{B}_k^G$ as a space of positive Hermitian matrix once a suitable basis of $H^0(X, L^k)$ is fixed. It is shown in [5] that I_k gives a quantization of the Aubin functional I. However in the extremal case, we need a modified version of the Aubin functional defined by the first author in order to feet with the balanced metrics. Let $V \in Lie(\mathrm{Aut}_0(X, L))$ and denote by $\sigma(t)$ the associated one parameter subgroup of $\mathrm{Aut}_0(X, L)$. Define up to a constant for each $\phi \in \mathcal{H}$ the function $\psi_{\sigma,\phi}$ by

$$\sigma(1)^*\omega_\phi = \omega_\phi + \sqrt{-1}\partial\overline{\partial}\psi_{\sigma,\phi}. \tag{5}$$

We will see in the sequel how to choose suitably a normalization constant for these potentials. We then consider a modified I functional defined up to a constant by its differential:

$$\delta I^\sigma(\phi)(\delta\phi) = \int_X \delta\phi(1 + \Delta_\phi)e^{\psi_{\sigma,\phi}}d\mu_\phi$$

where $\Delta_\phi = -g_\phi^{i\overline{j}}\frac{\partial}{\partial z_i}\frac{\partial}{\partial \overline{z}_j}$ is the complex Laplacian of g_ϕ. We will also need to consider the potentials ϕ as metrics on the tensor powers $L^{\otimes k}$, we thus consider the normalized vector fields $V_k = -\frac{V}{4k}$ and the associated one-parameter groups $\sigma_k(t)$. We choose the normalization

$$\int_X \exp\left(\psi_{\sigma_k,\phi}\right) d\mu_\phi = \frac{N_k}{k^n} \tag{6}$$

Then we define for each k

$$\delta I_k^\sigma(\phi)(\delta\phi) = \int_X k\delta\phi(1+\frac{\Delta_\phi}{k})e^{\psi_{\sigma_k,\phi}}k^n d\mu_\phi.$$

Remark 2.2.9. If σ is the identity, we recover the usual Aubin functional.

Remark 2.2.10. This one-form integrates along paths in $\mathcal{H}^G$ to a functional $I_k^\sigma(\phi)$ on $\mathcal{H}^G$, which is independent on the path used from 0 to ϕ. The proof of this fact is given in the Appendix, proposition 4.1.1.

We define $\mathcal{L}_k^\sigma$ on $\mathcal{H}^G$ and Z_k^σ on $\mathcal{B}_k^G$ by

$$\mathcal{L}_k^\sigma = I_k \circ Hilb_k + I_k^\sigma \tag{7}$$

and

$$Z_k^\sigma = I_k^\sigma \circ FS_k + I_k - k^n \log(k^n)V. \tag{8}$$

We will show in the following that these functionals converge to the modified K-energy in some sense. Note also that σ_k-balanced metrics are critical points for $\mathcal{L}_k^\sigma$ (proposition 3.1.3) and, if $FS_k(H_k)$ is a σ_k-balanced metric for some $H_k \in \mathcal{B}_k^G$, then H_k is a minimum for Z_k^σ (proposition 4.3.1).

3. Minima of the modified K-energy

The aim of this section is to prove Theorem A. For the convenience of the reader we give a sketch of the proof.

We will choose the special group G corresponding to the Killing field JV^* associated to the extremal vector field V^* of the extremal Kähler metric $\omega^* = \omega_{\phi^*}$. We know that the metrics $\omega_k^* = \omega + \sqrt{-1}\partial\overline{\partial}\phi_k^*$ with Kähler potentials $\phi_k^* = FS_k \circ Hilb_k(\phi^*)$ converge to ω^* ([26], [3] and [30]). We begin our proof by showing that the functionals $\mathcal{L}_k^\sigma$ converge to the modified Mabuchi functional on the space $\mathcal{H}^G$. Then we show that $Z_k^\sigma \circ Hilb_k$ and $\mathcal{L}_k^\sigma$ converge to the same functional, thus Z_k^σ gives a quantization of the modified Mabuchi functional and we reduce our problem to studying the minima of Z_k^σ. However the metrics ω_k^* constructed above are not in general critical points of Z_k^σ, as there is no reason for these metrics to be σ_k-balanced. We use instead an idea of Li [16] relying on the Bergman kernel expansion to show that these metrics ω_k^* are almost σ_k-balanced metrics, in the sense that $Hilb_k(\omega_k^*)$ is a minimum of the functional Z_k^σ up to an error which goes to zero when k tends to infinity.

Let V^* be the extremal vector field of the class $c_1(L)$. In the polarized case, the vector field JV^* generates a periodic action [12] by a one parameter-subgroup of automorphisms of (X, L). Fix G to be the one-parameter subgroup of $\mathrm{Aut}(X, L)$ associated to JV^*. This group is isomorphic to S^1 or trivial by the theorem of Futaki and Mabuchi [12]. This will be a group of isometries for each of our metrics.

Remark 3.0.11. The modified K-energy E^{G_m} is defined to be the modified Mabuchi functional with respect to a maximal compact connected subgroup G_m of $\mathrm{Aut}(X, L)$. Assume that G is contained in such a G_m. Then E^{G_m} is equal to E^G when restricted to the space of G_m-invariant potentials. Indeed, the projection of any G_m-invariant scalar curvature to the space of holomorphy potentials of $Lie(G_m)$ gives a potential for the extremal vector field by definition. Thus a minimum of E^G which is invariant under the G_m-action, such as an extremal metric, will be a minimum of the usual modified Mabuchi functional

Let σ_k be the element of $\mathrm{Aut}(X,L)$ associated to the vector field $-\frac{V^*}{4k}$. We will also need to define for each ϕ in $\mathcal{H}^G$ the function $\theta(\phi)$ to be the normalized (i.e. mean value zero) holomorphy potential of the vector field V^* with respect to the metric ω_ϕ:

$$g_\phi(V^*,\cdot) = d\theta(\phi)$$

or

$$\theta(\phi) = \Pi^G_\phi(S(\phi)).$$

3.1. **The functionals $\mathcal{L}^\sigma_k$ converge to E^G.** In this section we prove the following fact :

Proposition 3.1.1. *There are constants c_k such that*

$$\frac{2}{k^n}\mathcal{L}^\sigma_k + c_k \to E^G$$

as $k\to\infty$, where the convergence is uniform on $C^l(X,\mathbb{R})$ bounded subsets of $\mathcal{H}^G$.

Proof. We show that

$$\frac{2}{k^n}\delta\mathcal{L}^\sigma_k \to \delta E^G$$

uniformly on $C^l(X,\mathbb{R})$ bounded subsets of $\mathcal{H}^G$. First we compute $\delta\mathcal{L}^\sigma_k$. Following [10]:

$$\delta(I_k\circ Hilb_k)_\phi(\delta\phi) = -\int_X \delta\phi(\Delta_\phi + k)\rho_k(\phi)d\mu_\phi$$

and by definition

$$\delta(I^\sigma_k)_\phi(\delta\phi) = k^n\int_X \delta\phi(k+\Delta_\phi)e^{\psi_k(\phi)}d\mu_\phi$$

where we set $\psi_k(\cdot) = \psi_{\sigma_k,\cdot}$.
Then

$$\delta(\mathcal{L}^\sigma_k)_\phi(\delta\phi) = -\int_X \delta\phi(\Delta_\phi + k)(\rho_k(\phi) - k^n e^{\psi_k(\phi)})d\mu_\phi. \tag{9}$$

We need an expansion for the potential ψ_k :

$$\psi_k(\phi) = \frac{\theta(\phi)+\underline{S}}{2k} + \mathcal{O}_0(k^{-1})$$

which proof is postponed to lemma 3.1.2. Then by the expansions of $\psi_k(\phi)$ and $\rho_k(\phi)$

$$(\Delta_\phi+k)(\rho_k(\phi)-k^ne^{\psi_k(\phi)}) = k^n(\Delta_\phi+k)(1+\frac{S(\phi)}{2k}+\mathcal{O}(k^{-2})-1-\frac{\theta(\phi)+\underline{S}}{2k}+\mathcal{O}_0(k^{-1})),$$

$$(\Delta_\phi+k)(\rho_k(\phi)-k^ne^{\psi_k(\phi)}) = k^n(\frac{S(\phi)-\underline{S}-\theta(\phi))}{2}+\mathcal{O}(k^{-1})),$$

and

$$\frac{\delta(\mathcal{L}^\sigma_k)_\phi}{k^n} \to \frac{1}{2}\delta E^G_\phi.$$

As the expansions of $\psi_k(\phi)$ and $\rho_k(\phi)$ are uniform on bounded subsets of $C^l(X,\mathbb{R})$ the result follows. □

The following lemma will be useful :

Lemma 3.1.2. *The following expansion holds uniformly in $C^l(X,\mathbb{R})$ for $l >> 1$:*

$$\psi_k(\phi) = \frac{\theta(\phi) + \underline{S}}{2k} + \mathcal{O}_0(k^{-1}) \tag{10}$$

where $\mathcal{O}_0(k^{-1})$ denotes k^{-1}-times a function $\varepsilon(k)$ on X with $\varepsilon(k) \to 0$ in $C^l(X,\mathbb{R})$ as $k \to 0$.

Proof. By definition

$$\sigma_k(1)^*\omega(\phi) - \omega(\phi) = \sqrt{-1}\partial\overline{\partial}\psi_k(\phi),$$

then

$$\sigma_1(\frac{1}{k})^*\omega(\phi) - \omega(\phi) = \sqrt{-1}\partial\overline{\partial}\psi_k(\phi),$$

where $\sigma_1(\frac{1}{k})$ is equal to $\exp(-\frac{1}{4k}V^*)$. Dividing by $\dfrac{1}{k}$, and letting k go to infinity,

$$\mathcal{L}_{-\frac{1}{4}V^*}\omega(\phi) = \sqrt{-1}\partial\overline{\partial}\lim_{k\to\infty}(k\psi_k(\phi))$$

Then by Cartan's formula,

$$\begin{aligned}\mathcal{L}_{-\frac{1}{4}V^*}\omega(\phi) &= -\tfrac{1}{4}d\omega_\phi(V^*,\cdot)\\ &= -\tfrac{1}{4}dg_\phi(V^*,J\cdot)\end{aligned}$$

and by definition of holomorphy potentials

$$\mathcal{L}_{-\frac{1}{4}V^*}\omega(\phi) = -\frac{1}{4}dd^c\theta(\phi) = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\theta(\phi)$$

thus

$$\lim_{k\to\infty}(k\psi_k(\phi)) = \frac{\theta(\phi)+c}{2}$$

for some constant c. By the normalization (6) of the function $\psi_k(\phi)$ we deduce

$$\frac{N_k}{k^n} = \int_X \exp\left(\psi_{\sigma_k,\phi}\right)d\mu_\phi = \int_X 1 + \frac{\theta(\phi)+c}{2k} + \mathcal{O}(k^{-2})d\mu_\phi.$$

Recall that we choose $\theta(\phi)$ normalized to have mean value zero. Using formula (2) to expand $N_k = dim(H^0(X,L^k))$, we conclude that $c = \underline{S}$. □

From the above computations we also deduce the following :

Proposition 3.1.3. *Let $\phi \in \mathcal{H}$ be a k^{th} σ_k-balanced metric. Then ϕ is a critical point of $\mathcal{L}_k^\sigma$.*

Proof. By equation (4) of σ_k-balanced metrics and by definition (5) of $\psi_k(\phi)$ we deduce

$$\rho_k(\phi) = C\exp(\psi_k(\phi))$$

for some constant C. Integrating over X and using the expansions (2) and (10) we deduce

$$\rho_k(\phi) = k^n\exp(\psi_k(\phi)).$$

The result follows from the computation of the differential of $\mathcal{L}_k^\sigma$, equation (9). □

A direct computation implies the similar result for Z_k^σ (see proposition 4.3.1 in the appendix).

3.2. **Comparison of Z_k^σ and $\mathcal{L}_k^\sigma$.** The aim of this section is to show that $Z_k^\sigma \circ Hilb_k$ and $\mathcal{L}_k^\sigma$ converge to the same functional. We will need the two following lemmas:

Lemma 3.2.1. *The second derivative of I_k^σ along a path $\phi_s \in \mathcal{H}^G$ is equal to*

$$\frac{d^2}{ds^2} I_k^\sigma(\phi_s) = k^n \int_X (\phi'' - \frac{1}{2}|d\phi'|^2)(k + \Delta_{\phi_s}) e^{\psi_k(\phi_s)} d\mu_{\phi_s}$$

Proof. The proof of this result is given in the Appendix, section 4.2. □

Lemma 3.2.2. *Let $\phi \in \mathcal{H}^G$. Then there exists an integer k_0, depending on ϕ, such that for each $k \geq k_0$, the functional I_k^σ is concave along the path*

$$\begin{array}{ccc} [0,1] & \to & \mathcal{H}^G \\ s & \mapsto & \phi + \frac{s}{k}\log(\rho_k(\phi)) \end{array}$$

Proof. By lemma 3.2.1, the second derivative of I_k^σ along the path $\phi_k(s) = \phi + \frac{s}{k}\log(\rho_k(\phi))$ is

$$k^n \int_X (\phi_k'' - \frac{1}{2}|d\phi_k'|^2)(k + \Delta_{\phi_k(s)}) e^{\psi_k(\phi_k(s))} d\mu_{\phi_k(s)}.$$

As $\phi_k' = \frac{1}{k}\log(\rho_k(\phi))$ and $\phi_k'' = 0$, this expression simplifies:

$$\frac{d^2}{ds^2} I_k^\sigma(\phi_k(s)) = -k^n \int_X \frac{1}{2}|d\frac{1}{k}\log(\rho_k(\phi))|^2 (k + \Delta_{\phi_k(s)}) e^{\psi_k(\phi_k(s))} d\mu_{\phi_k(s)}.$$

We compute the leading term in the above expression as k goes to infinity. To simplify notation, let $T_k(\phi) = FS_k \circ Hilb_k(\phi)$. Note that $\omega_{\phi_1} = \omega_{T_k(\phi)}$. From (1), the difference between ω_{ϕ_0} and ω_{ϕ_1} is

$$\omega_{\phi_0} - \omega_{\phi_1} = \mathcal{O}(k^{-2}).$$

Thus we have the estimates

$$\Delta_{\phi_k(s)} = \Delta_\phi + \mathcal{O}(k^{-1}),$$

$$d\mu_{\phi_k(s)} = d\mu_\phi + \mathcal{O}(k^{-1})$$

and

$$\psi_k(\phi_k(s)) = \psi_k(\phi) + \mathcal{O}(k^{-1}).$$

Then

$$\frac{d^2}{ds^2} I_k^\sigma(\phi_k(s)) = -k^n \int_X \frac{1}{2}|d\frac{1}{k}\log(\rho_k(\phi))|^2 (k + \Delta_\phi) e^{\psi_k(\phi)} d\mu_\phi + \mathcal{O}(k^{n-1}).$$

From this we deduce that the leading term as k tends to infinity is

$$-\frac{k^{n-1}}{2} \int_X |dS(\phi)|^2 d\mu_\phi < 0$$

where once again we used the expansions of Bergman kernel and of $\psi_k(\phi)$ from lemma 3.1.2. □

Now we can prove the main result of this section:

Proposition 3.2.3. *For each potential $\phi \in \mathcal{H}^G$, we have*

$$\lim_{k\to\infty} k^{-n}(\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)) = 0$$

Proof. By definition,

$$k^{-n}(\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)) = k^{-n}(I_k^\sigma(T_k(\phi)) - I_k^\sigma(\phi) - k^n \log(k^n)V)$$

where $T_k = FS_k \circ Hilb_k$. From lemma 3.2.2, for k large enough, the functional I_k^σ is concave along the path

$$s \mapsto \phi + \frac{s}{k}\log(\rho_k(\phi))$$

going from ϕ to $T_k(\phi)$ in $\mathcal{H}^G$.
Thus

$$(\delta I_k^\sigma)_\phi(\frac{1}{k}\log\rho_k(\phi)) \geq (I_k^\sigma(T_k(\phi)) - I_k^\sigma(\phi)) \geq (\delta I_k^\sigma)_{T_k(\phi)}(\frac{1}{k}\log\rho_k(\phi)). \tag{11}$$

We deduce from the definitions that

$$k^{-n}(\delta I_k^\sigma)_\phi(\frac{1}{k}\log\rho_k(\phi)) - \log(k^n)V \geq k^{-n}(\mathcal{L}_k^G(\phi) - Z_k^G \circ Hilb_k(\phi)) \tag{12}$$

and

$$k^{-n}(\mathcal{L}_k^G(\phi) - Z_k^G \circ Hilb_k(\phi)) \geq k^{-n}(\delta I_k^\sigma)_{T_k(\phi)}(\frac{1}{k}\log\rho_k(\phi)) - \log(k^n)V \tag{13}$$

and it remains to show that the left hand side of (12) and the right hand side of (13) tend to zero. First

$$k^{-n}(\delta I_k^\sigma)_\phi(\frac{1}{k}\log\rho_k(\phi)) - \log(k^n)V = \int_X (\frac{1}{k}\log(\rho_k(\phi)))(k+\Delta_\phi)e^{\psi_k(\phi)}d\mu_\phi - V\log(k^n)$$

$$= \int_X (\log(k^n) + \frac{S(\phi)}{2k} + \mathcal{O}(k^{-2}))(1+\frac{\Delta_\phi}{k})(1+\frac{\theta(\phi)+\underline{S}}{2k} + \mathcal{O}_0(k^{-1}))d\mu_\phi - V\log(k^n)$$

by the expansion of Bergman kernel and lemma 3.1.2. If follows that

$$k^{-n}(\delta I_k^\sigma)_\phi(\frac{1}{k}\log\rho_k(\phi)) - \log(k^n)V = V\log(k^n) + \mathcal{O}(k^{-1}) - V\log(k^n) \to 0$$

as $k \to \infty$.

Note that we didn't make use of the fact that the derivative δI_k^σ was evaluated at ϕ, so the above argument extends to the last term of the inequality (13), evaluated at $T_k(\phi)$, which thus tends to zero as well. This ends the proof. □

3.3. The metrics $Hilb_k(\omega^*)$ are almost σ-balanced.

We will need the following convexity property of Z_k^σ:

Lemma 3.3.1. *The functional Z_k^σ is convex along geodesics in $\mathcal{B}_k^G$.*

Proof. Here we abbreviate the subscript k. Take a geodesic $\{H(s)\}_{s\in\mathbb{R}}$ in $\mathcal{B}^G$. By choosing an appropriate orthonormal basis $\{\tau_\alpha\}$ of $H(0)$, $H(s)$ is represented by

$$H(s) = diag(e^{2\lambda_\alpha s}),\ \ \lambda_\alpha \in \mathbb{R}.$$

Let

$$\phi_s = \log\big(\sum_\alpha e^{\lambda_\alpha s}|\tau_\alpha|^2 / \sum_\beta |\tau_\beta|^2\big).$$

According to Proposition 1 in [10] (also [22]), the inequality

$$\int_X (\phi_s'' - \frac{1}{2}|d\phi_s'|^2)d\mu_\phi = \sum_\alpha \int_X |(\nabla\phi_s', \nabla\tau_\alpha) - (\lambda_\alpha - \phi_s')\tau_\alpha|^2 d\mu_\phi \geq 0 \tag{14}$$

holds. For any C^∞-section f and any holomorphic section τ of L, we have

$$\begin{aligned}
&\int_X |f|^2\Delta_\phi|s|^2 d\mu \\
= \;& \frac{1}{2}\Big(\int_X \{(\nabla f, f) + (f, \overline{\nabla} f)\}(s, \nabla s) d\mu \\
&+ \int_X \{(\overline{\nabla} f, f) + (f, \nabla f)\}(\nabla s, s) d\mu\Big) \\
= \;& \frac{1}{4}\int_X \big\{(|(\nabla f, f)|^2 + |(s, \nabla s)|^2 - |(\nabla f, f) - (\nabla s, s)|^2)\big\} \\
&+ \big\{(|(f, \overline{\nabla} f)|^2 + |(\nabla s, s)|^2 - |(f, \overline{\nabla} f) - (\nabla s, s)|^2)\big\} d\mu \geq 0. \qquad (15)
\end{aligned}$$

Hence, (14), (15) and Lemma 3.2.1 complete the proof. □

Proposition 3.3.2. *Let $\phi \in \mathcal{H}^G$. Then there are functions $\varepsilon_\phi(k)$ such that*

$$k^{-n}(Z_k^\sigma \circ Hilb_k(\phi)) \geq k^{-n}(Z_k^\sigma \circ Hilb_k(\phi^*)) + \varepsilon_\phi(k)$$

and such that $\lim_{k\to\infty} \varepsilon_\phi(k) = 0$ in $C^l(X, \mathbb{R})$ for $l >> 1$.

Proof. We follow Li's proof of [16][Lemma 3.3.], adapted to our more general setting. In the sequel, C will stand for a constant depending on ϕ, ϕ^* and the volume of the polarized manifold (X, L), but independent on k. The precise value of this constant might change but it won't be important for us.

Let's set $H_k^* = Hilb_k(\phi^*)$ and $H_k = Hilb_k(\phi)$. We choose an orthonormal basis $\{\tau_\alpha^{(k)}\}$ of H_k^* such that in this basis H_k^* is represented by the identity and

$$H_k = diag(e^{2\lambda_\alpha^{(k)}}).$$

Then evaluating H_k on the orthonormal vectors $e^{\lambda_\alpha^{(k)}}\tau_\alpha^k$:

$$e^{-2\lambda_\alpha^{(k)}} = \int_X |\tau_\alpha^k|_{h_0^k} d\mu_0. \qquad (16)$$

Comparing the metrics we have the existence of $C > 0$ such that

$$h_0^k \leq C^k h_{\phi^*}^k$$

from which we deduce with (16) the following estimate:

$$|\lambda_\alpha^k| \leq Ck. \qquad (17)$$

Let's consider the one-parameter subgroup of $\mathcal{B}_k^G$:

$$s \mapsto H_k(s) = diag(e^{2s\lambda_\alpha^{(k)}}).$$

This is a geodesic that goes from H_k^* to H_k in $\mathcal{B}_k^G$, thus by lemma 3.3.1:

$$k^{-n}(Z_k^\sigma(H_k) - Z_k^\sigma(H_k^*)) \geq k^{-n} f_k'(0)$$

with

$$f_k(s) = Z_k^\sigma(H_k(s)).$$

We then need to show that $\lim_{k\to\infty} k^{-n} f_k'(0) = 0$. By a straightforward computation

$$k^{-n} f_k'(0) = 2k^{-n}\sum_\alpha \lambda_\alpha^{(k)} - \frac{2}{k}\int_X \frac{\rho_k^\lambda}{\rho_k}(k + \Delta)e^{\psi_k} d\mu$$

where $\rho_k^\lambda = \sum_\alpha \lambda_\alpha^{(k)} |\tau_\alpha^{(k)}|^2_{h_0^k}$ and the quantities ρ_k, Δ, ψ_k and $d\mu$ are computed with respect to the extremal metric ω_{ϕ^*}. Then

$$2^{-1}k^{-n}f_k'(0) = k^{-n}\sum_\alpha \lambda_\alpha^{(k)} - \int_X \frac{\rho_k^\lambda}{\rho_k} e^{\psi_k} d\mu - \frac{1}{k}\int_X \frac{\rho_k^\lambda}{\rho_k}\Delta e^{\psi_k} d\mu. \tag{18}$$

We first show that the last term in the sum of (18) tends to zero. First note that from (17),

$$|\frac{\rho_k^\lambda}{\rho_k}| \leq Ck$$

thus

$$|\frac{1}{k}\int_X \frac{\rho_k^\lambda}{\rho_k}\Delta e^{\psi_k} d\mu| \leq C\int_X |\Delta e^{\psi_k}| d\mu$$

and using lemma 3.1.2 we deduce that this term goes to zero as k tends to infinity. Then consider the second term in the right hand side of equation (18). Using the expansions of ψ_k and ρ_k we deduce:

$$\rho_k^{-1} e^{\psi_k} = k^{-n}(1 - \frac{S}{2k} + \mathcal{O}(k^{-2}))(1 + \frac{\theta + \underline{S}}{2k} + \mathcal{O}_0(k^{-1})).$$

Here we use our crucial assumption, that is ω_{ϕ^*} is extremal, so $S = \theta + \underline{S}$ and thus

$$\rho_k^{-1} e^{\psi_k} = k^{-n}(1 + \mathcal{O}_0(k^{-1})).$$

Then

$$\int_X \frac{\rho_k^\lambda}{\rho_k} e^{\psi_k} d\mu = \int_X \frac{\rho_k^\lambda}{k^n}(1 + \mathcal{O}_0(k^{-1}))d\mu.$$

As

$$\int_X \frac{\rho_k^\lambda}{k^n} d\mu = k^{-n}\sum_\alpha \lambda_\alpha^{(k)},$$

the only remaining term to control at infinity in $k^{-n}f_k'(0)$ is

$$\int_X \frac{\rho_k^\lambda}{k^n}\mathcal{O}_0(k^{-1})d\mu.$$

Using (17),

$$|\frac{\rho_k^\lambda}{k^n}\mathcal{O}_0(k^{-1})| \leq CkN_k k^{-n}|\mathcal{O}_0(k^{-1})|.$$

By equation (2), $N_k k^{-n}$ is bounded and as $\mathcal{O}_0(k^{-1}) = k^{-1}\epsilon(k)$ with $\epsilon(k) \to 0$

$$\lim_{k\to\infty}\int_X \frac{\rho_k^\lambda}{k^n}\mathcal{O}_0(k^{-1})d\mu = 0$$

and

$$\lim_{k\to\infty} k^{-n}f_k'(0) = 0.$$

$\square$

3.4. **Conclusion, proof of theorem A.** We conclude this section with the proof of Theorem A. We show the following stronger theorem, which implies theorem A with remark 3.0.11:

Theorem 3.4.1. *Let (X, L) be a polarized manifold that carries extremal metrics representing $c_1(L)$. The modified Mabuchi functional with respect to the G-action induced by the extremal vector field of $c_1(L)$ attains its minimum at the extremal metrics.*

Proof. Let $\phi \in \mathcal{H}^G$ and ϕ^* be the potential of an extremal metric.

$$\mathcal{L}_k^\sigma(\phi) \quad = \quad Z_k^\sigma \circ Hilb_k(\phi) + (\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)). \tag{19}$$

By proposition 3.3.2:

$$\mathcal{L}_k^\sigma(\phi) \quad \geq \quad Z_k^\sigma \circ Hilb_k(\phi^*) + k^n \varepsilon_\phi(k) + (\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)) \tag{20}$$

Then

$$\begin{aligned} \mathcal{L}_k^\sigma(\phi) \quad \geq \quad & \mathcal{L}_k^\sigma(\phi^*) + (Z_k^\sigma \circ Hilb_k(\phi^*) - \mathcal{L}_k^\sigma(\phi^*)) + \\ & k^n \varepsilon_\phi(k) + (\mathcal{L}_k^\sigma(\phi) - Z_k^\sigma \circ Hilb_k(\phi)) \end{aligned} \tag{21}$$

To conclude, from proposition 3.2.3,

$$k^{-n}(Z_k^\sigma \circ Hilb_k(\phi^*) - \mathcal{L}_k^\sigma(\phi^*)) \to 0$$

and

$$k^{-n}(Z_k^\sigma \circ Hilb_k(\phi) - \mathcal{L}_k^\sigma(\phi)) \to 0$$

as k tends to infinity. So does $\varepsilon_\phi(k)$ by construction, see proposition 3.3.2. Thus the result follows from proposition 3.1.1, multiplying by k^{-n} and letting k go to infinity in (21). □

4. Appendix

We give the proof of the results concerning the σ-balanced metrics. We denote by $(\cdot, \cdot)$ any of the following Hermitian pairings

$$\begin{array}{ll} T^*X \times (T^*X \times L) \to L, & L \times (T^*X \times L) \to T^*X, \\ L \times L \to \mathbb{C}, & T^*X \times T^*X \to \mathbb{C} \end{array}$$

obtained by $\phi \in \mathcal{H}$ and ω_ϕ. We denote the connection of type $(1, 0)$ on the holomorphic tangent bundle $T'X$ by ∇.

4.1. The definition of I^σ.

Proposition 4.1.1. *$I^\sigma(\phi)$ is independent of the choice of a path from 0 to ϕ.*

Proof. Since $I^\sigma(\phi)$ satisfies the cocycle property

$$I^\sigma(\phi_1, \phi_3) = I^\sigma(\phi_1, \phi_2) + I^\sigma(\phi_2, \phi_3)$$

by definition, it is sufficient to prove $\frac{\partial^2}{\partial s \partial t} I^\sigma(\phi_{0,0}, \phi_{t,s})$ is symmetric with respect to s and t for any family of path

$$\{\Phi = \phi_{t,s} \mid (s,t) \in [0,1] \times [0,1],\, \phi_{0,s} = \phi_{1,s} \equiv 0\}$$

in $\mathcal{H}$.

$$\frac{\partial^2}{\partial s\partial t}I^\sigma(\phi_{0,0},\phi_{t,s}) = \frac{\partial}{\partial s}\int_X \big((1+\Delta_\Phi)\frac{\partial\Phi}{\partial t}\big)e^{\psi_{\sigma,\Phi}}d\mu_\Phi$$

$$= \int_X \big((\frac{\partial}{\partial s}\Delta_\Phi)\frac{\partial\Phi}{\partial t}\big)e^{\psi_{\sigma,\Phi}}d\mu_\Phi + \int_X \big((1+\Delta_\Phi)\frac{\partial^2\Phi}{\partial s\partial t}\big)e^{\psi_{\sigma,\Phi}}d\mu_\Phi$$

$$+\int_X \big((1+\Delta_\Phi)\frac{\partial\Phi}{\partial t}\big)\big(\frac{\partial e^{\psi_{\sigma,\Phi}}}{\partial s}\big)d\mu_\Phi - \int_X \big((1+\Delta_\Phi)\frac{\partial\Phi}{\partial t}\big)e^{\psi_{\sigma,\Phi}}\big(\Delta_\Phi\frac{\partial\Phi}{\partial s}\big)d\mu_\Phi. \tag{22}$$

The first term in (22) is

$$\int_X (\nabla\overline{\nabla}\frac{\partial\Phi}{\partial t},\nabla\overline{\nabla}\frac{\partial\Phi}{\partial s})e^{\psi_{\sigma,\Phi}}d\mu_\Phi$$

which is symmetric. The second term is obviously symmetric. The third term is

$$\int_X \frac{\partial\Phi}{\partial t}(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s})e^{\psi_{\sigma,\Phi}}d\mu_\Phi + \int_X \big(\Delta_\Phi\frac{\partial\Phi}{\partial t}\big)\big(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s}\big)e^{\psi_{\sigma,\Phi}}d\mu_\Phi. \tag{23}$$

Here we use the following equality.

Lemma 4.1.2.

$$\frac{\partial\psi_{\sigma,\Phi}}{\partial s} = \big(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s}\big). \tag{24}$$

Proof. Let v be the gradient vector field of $\frac{\partial\Phi}{\partial s}$, i.e.,

$$v = grad_{\omega_\Phi}\left(\frac{\partial\Phi}{\partial s}\right) = \sum_{i,j} g^{i\bar{j}}\frac{\partial}{\partial\bar{z}^j}\left(\frac{\partial\Phi}{\partial s}\right)\frac{\partial}{\partial z^i}. \tag{25}$$

We have

$$\begin{aligned}\frac{\partial}{\partial s}(\sigma(1)^*\omega_\Phi-\omega_\Phi) &= L_v(\sigma(1)^*\omega_\Phi-\omega_\Phi) = \frac{\sqrt{-1}}{2\pi}d\iota_v\partial\bar{\partial}\psi_{\sigma,\Phi}\\ &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\big(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s}\big)\end{aligned}$$

where L_v is the Lie derivative along v. Then, there exists some constant c such that

$$\frac{\partial\psi_{\sigma,\Phi}}{\partial s} = \big(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s}\big)+c. \tag{26}$$

Recall that

$$\int_X \psi_{\sigma,\Phi}d\mu_\Phi$$

is constant with respect to s, t by normalization of $\psi_{\sigma,\Phi}$. Since

$$0 = \frac{\partial}{\partial s}\int_X \psi_{\sigma,\Phi}d\mu_\Phi = \int_X\left(\frac{\partial\psi_{\sigma,\Phi}}{\partial s}-\big(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s}\big)\right)d\mu_\Phi,$$

the constant c in (26) is zero. Hence, (24) is proved. □

The forth term is

$$-\int_X e^{\psi_{\sigma,\Phi}}\frac{\partial\Phi}{\partial t}\Delta_\Phi\frac{\partial\Phi}{\partial s}d\mu_\Phi - \int_X e^{\psi_{\sigma,\Phi}}\Delta_\Phi\frac{\partial\Phi}{\partial t}\Delta_\Phi\frac{\partial\Phi}{\partial s}d\mu_\Phi. \tag{27}$$

The sum of the first term in (23) and the first term in (27) is

$$-\int_X \frac{\partial\Phi}{\partial t}\left(\Delta_\Phi\frac{\partial\Phi}{\partial s}+\big(\nabla\psi_{\sigma,\Phi},\nabla\frac{\partial\Phi}{\partial s}\big)\right)e^{\psi_{\sigma,\Phi}}d\mu_\Phi.$$

This is symmetric, because the operator $\Delta_\Phi + (\nabla\psi_{\sigma,\Phi}, \nabla)$ is self-adjoint with respect to the weighted volume form $e^{\psi_{\sigma,\Phi}} d\mu_\Phi$. The remaining is the second term in (23). It is

$$-\int_X \big(\nabla\overline{\nabla}\psi_{\sigma,\Phi}, \nabla\frac{\partial\Phi}{\partial t}\overline{\nabla}\frac{\partial\Phi}{\partial s}\big)e^{\psi_{\sigma,\Phi}}d\mu_\Phi - \int_X \big(\nabla\frac{\partial\Phi}{\partial t}, \nabla\psi_{\sigma,\Phi}\big)\big(\nabla\frac{\partial\Phi}{\partial s}, \nabla\psi_{\sigma,\Phi}\big)e^{\psi_{\sigma,\Phi}}d\mu_\Phi,$$

which is symmetric. □

4.2. **Second derivative of I_k^σ.** We give a computation of the second derivative of I_k^σ.

Proof of Lemma 3.2.1.

$$\begin{aligned}
V\frac{d^2}{ds^2}I_k^\sigma(\phi_s) &= k^n\frac{d}{ds}\int_X (k+\Delta_\phi)\phi' e^{\psi_{\sigma,\phi}}d\mu_\phi \\
&= k^n\int_X (\nabla\overline{\nabla}\phi', \nabla\overline{\nabla}\phi')e^{\psi_{\sigma,\phi}}d\mu_\phi + k^n\int_X (k+\Delta_\phi)\phi'' e^{\psi_{\sigma,\phi}}d\mu_\phi \\
&\quad + k^n\int_X ((k+\Delta_\phi)\phi')\psi'_{\sigma,\phi}e^{\psi_{\sigma,\phi}}d\mu_\phi - k^n\int_X ((k+\Delta_\phi)\phi')e^{\psi_{\sigma,\phi}}\Delta_\phi\phi' d\mu_\phi. \qquad (28)
\end{aligned}$$

From (24), the third term in (28) is equal to

$$k^n\int_X ((k+\Delta_\phi)\phi')(\nabla\psi_{\sigma,\phi}, \nabla\phi')e^{\psi_{\sigma,\phi}}d\mu_\phi. \qquad (29)$$

By the partial integral, the forth term in (28) is equal to

$$\begin{aligned}
&-k^{n+1}\int_X |\nabla\phi'|^2 e^{\psi_{\sigma,\phi}}d\mu_\phi - k^{n+1}\int_X \phi' e^{\psi_{\sigma,\phi}}(\nabla\psi_{\sigma,\phi}, \nabla\phi')d\mu_\phi \\
&-k^n\int_X (\nabla\Delta_\phi\phi', \nabla\phi')e^{\psi_{\sigma,\phi}}d\mu_\phi - k^n\int_X (\Delta_\phi\phi')(\nabla\psi_{\sigma,\phi}, \nabla\phi')e^{\psi_{\sigma,\phi}}d\mu_\phi. \qquad (30)
\end{aligned}$$

Remark that the sum of the second and forth terms in (30) cancels (29). The third term in (30) is

$$\begin{aligned}
&-k^n\int_X (\nabla\overline{\nabla}\phi', \nabla\overline{\nabla}\phi')e^{\psi_{\sigma,\phi}}d\mu_\phi - k^n\int_X (\nabla\overline{\nabla}\phi', \nabla\psi_{\sigma,\phi}\overline{\nabla}\phi')e^{\psi_{\sigma,\phi}}d\mu_\phi \\
=\ &-k^n\int_X (\nabla\overline{\nabla}\phi', \nabla\overline{\nabla}\phi')e^{\psi_{\sigma,\phi}}d\mu_\phi - k^n\int_X |\nabla\phi'|^2\Delta_\phi\psi_{\sigma,\phi}e^{\psi_{\sigma,\phi}}d\mu_\phi \\
&+k^n\int_X |\nabla\phi'|^2|\nabla\psi_{\sigma,\phi}|^2 e^{\psi_{\sigma,\phi}}d\mu_\phi \\
=\ &-k^n\int_X (\nabla\overline{\nabla}\phi', \nabla\overline{\nabla}\phi')e^{\psi_{\sigma,\phi}}d\mu_\phi - k^n\int_X |\nabla\phi'|^2\Delta_\phi e^{\psi_{\sigma,\phi}}d\mu_\phi. \qquad (31)
\end{aligned}$$

Substituting (29), (30) and (31) for (28), we get the second derivative of $I_k^\sigma(\phi)$. □

4.3. **Minima of Z_k^σ.** The following fact is fundamental to understand the idea of this paper, although we do not use as it stands in the proof of the main theorem. We give a proof for the convenience of the reader.

Proposition 4.3.1. *If $FS_k(H_k)$ is a σ_k-balanced metric for some $H_k \in \mathcal{B}_k^G$, then H_k is a minimum of Z_k^σ on $\mathcal{B}_k^G$.*

Proof. Here we abbreviate the subscript k. In Lemma 3.3.1, we show the convexity of Z^σ along geodesics in $\mathcal{B}^G$. Then, it is sufficient to show that the σ-balanced point $H \in \mathcal{B}^G$ is a critical points of Z^σ. Take any variation $\delta H = \frac{d}{dt}\big|_{t=0}\tau(t)$ at H, where $\tau(t) \in SL(H^0(M,L))$. Diagonalizing with respect to some orthonormal basis $\{s\}_\alpha$, $\tau(t)$ is represented by the diagonal matrix

$$\tau(t) = diag(e^{\lambda_\alpha t}), \quad \sum_\alpha \lambda_\alpha = 0,\ \lambda_\alpha \in \mathbb{R}.$$

Then, the variation of the Kähler potential $\varphi_t = FS(\lambda(t)\cdot H)$ at $t=0$ is given by

$$\varphi_0' = \frac{\sum_\alpha \lambda_\alpha |s_\alpha|^2}{\sum_\beta |s_\beta|^2}.$$

From this and (24), we have

$$\begin{aligned}
\delta Z^\sigma(\delta H) &= \int_X \varphi_0'(1+\Delta_{FS(H)})e^{\psi_{\sigma,FS(H)}}d\mu_{FS(H)} \\
&= \int_X \varphi_0' e^{\psi_{\sigma,FS(H)}} + \frac{d}{dt}\bigg|_{t=0} e^{\psi_{\sigma,FS(\lambda(t)\cdot H)}}d\mu_{FS(H)} \\
&= \int_X \frac{\sum_\alpha \lambda_\alpha |s_\alpha|^2}{\sum_\beta |s_\beta|^2}\left(\frac{\sum_\gamma |\sigma^* s_\gamma|^2}{\sum_\beta |s_\beta|^2}\right) + \left\{\frac{d}{dt}\bigg|_{t=0}\left(\frac{\sum_\gamma |\big(\lambda(t)\sigma\big)^* s_\gamma|^2}{\sum_\beta |\lambda(t)^* s_\beta|^2}\right)\right\}d\mu_{FS(H)} \\
&= \int_X \frac{\sum_\alpha \lambda_\alpha |\sigma^* s_\alpha|^2}{\sum_\beta |s_\beta|^2}d\mu_{FS(H)}.
\end{aligned}$$

Since H is σ-balanced, $\{c(\sigma^* s_\alpha)\}_\alpha$ is an orthonormal basis with respect to $T(H)$ for some $c>0$. Therefore, we have $\delta Z^\sigma(\delta H) = 0$. The proof is completed. □

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